

GEOMETRIC PROOF OF A CONJECTURE OF FULTON

PRAKASH BELKALE

ABSTRACT. We give a geometric proof of a conjecture of Fulton on the multiplicities of irreducible representations in a tensor product of irreducible representations for $GL(r)$. This conjecture was proven earlier by Knutson, Tao and Woodward using the Honeycomb theory.

1. INTRODUCTION

Recall that irreducible *polynomial* representations of $GL(r)$ are indexed by sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0) \in \mathbb{Z}^r$. Denote the representation corresponding to λ by V_λ . Define Littlewood-Richardson coefficients $c_{\mu,\nu}^\lambda$ by: $V_\mu \otimes V_\nu = \sum c_{\mu,\nu}^\lambda V_\lambda$. W. Fulton conjectured that for any positive integer N ,

$$c_{\mu,\nu}^\lambda = 1 \Leftrightarrow c_{N\mu, N\nu}^{N\lambda} = 1.$$

This conjecture was proved by A. Knutson, T. Tao and C. Woodward [KTW] using the Honeycomb theory.

In this article we give a geometric proof of Fulton's conjecture based the geometric proof of Horn and saturation conjectures given in [GH]. The techniques in the proof to be given can be applied in quantum cohomology (this is our main motivation, see [QH] for the multiplicative generalization of Horn and saturation conjectures), and hopefully also in quiver theory, to prove analogues of Fulton's conjecture.

Our proof deduces Fulton's conjecture from the projectivity of some Geometric invariant theory (GIT) moduli spaces, a technique which is sufficiently categorical for generalizations. This technique is most easily understood in the geometric proof of Fulton's original conjecture given here.

I thank Harm Derksen for useful discussions.

1.1. Conventions. We make the following conventions:

- An integer $s \geq 1$ will be fixed for the proof.
- For a vector space W , let $\text{Fl}(W)$ denote the variety of complete flags on it. If $\mathcal{E} \in \text{Fl}(W)^s$, we will assume that \mathcal{E} is written in the form $(E_\bullet^1, \dots, E_\bullet^s)$.
- We use the notation $[n] = \{1, \dots, n\}$.

2. SOME RESULTS AND NOTATION FROM [GH]

In this section we recall some results and notation from [GH]. The reader may wish to turn to Section 3 now.

2.1. Schubert cells in Grassmannians. Let $I \subseteq [n] = \{1, \dots, n\}$ be a subset of cardinality r . Such a set will always be written as $I = \{i_1 < \dots < i_r\}$. Let

$$E_\bullet : \{0\} = E_0 \subsetneq E_1 \subsetneq \dots \subsetneq E_n = W$$

be a complete flag in an n -dimensional vector space W . Define the Schubert cell $\Omega_I^o(E_\bullet) \subseteq \text{Gr}(r, W)$ by

$$\Omega_I^o(E_\bullet) = \{V \in \text{Gr}(r, W) \mid \text{rk}(V \cap E_u) = a \text{ for } i_a \leq u < i_{a+1}, a = 0, \dots, r\}$$

where i_0 is defined to be 0 and $i_{r+1} = n$. $\Omega_I^o(E_\bullet)$ is smooth. Its closure will be denoted by $\Omega_I(E_\bullet)$. For a fixed complete flag on W , it is easy to see that ([F1], §1) every r -dimensional vector subspace belongs to a unique Schubert cell.

Definition 2.1. Let V be a r -dimensional subspace of an n -dimensional vector space W , and $\mathcal{E} \in \text{Fl}(W)^s$. Let I^1, \dots, I^s be the unique subsets of $[n]$ each of cardinality r such that $V \in \Omega_{I^j}^o(E_\bullet^j)$ for $j = 1, \dots, s$. Define $\dim(V, W, \mathcal{E})$ to be the expected dimension of the intersection $\cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$. That is,

$$\begin{aligned} \dim(V, W, \mathcal{E}) &= \dim(\text{Gr}(r, n)) - \sum_{j=1}^s \text{codim}(\omega_{I^j}) \\ &= r(n - r) - \sum_{j=1}^s \sum_{a=1}^r (n - r + a - i_a^j). \end{aligned}$$

2.2. Induced flags. Suppose that W is an n -dimensional vector space and $V \subseteq W$ an r -dimensional subspace. Let E_\bullet be a complete flag on W . This induces a complete flag on V and a complete flag on W/V by intersecting E_\bullet with V and by projection $p : W \rightarrow W/V$ respectively. We denote these by $E_\bullet(V)$ and $E_\bullet(W/V)$ respectively. Explicitly, if $V \in \Omega_I^o(E_\bullet)$ and $[n] \setminus I = \{\alpha(1) < \dots < \alpha(n - r)\}$, then $E_a(V) = E_{i_a} \cap V$, $a = 1, \dots, r$ and $E_b(W/V) = p(E_{\alpha(b)})$, $b = 1, \dots, n - r$. Given an ordered collection of flags $\mathcal{E} \in \text{Fl}(W)^s$ we obtain ordered collections of flags $\mathcal{E}(V) \in \text{Fl}(V)^s$ and $\mathcal{E}(W/V) \in \text{Fl}(W/V)^s$ by performing the above operations in each coordinate factor.

The following lemma follows from a direct calculation (see [F1], Lemma 2 (i)).

Lemma 2.2. Let W be an n -dimensional vector space. Suppose $E_\bullet \in \text{Fl}(W)$ and $S \subseteq V \subseteq W$ are subspaces with $\text{rk}(V) = r$ and $\text{rk}(S) = d$. Let I be the unique subset of $[n]$ of cardinality r such that $V \in \Omega_I^o(E_\bullet) \subseteq \text{Gr}(r, W)$, and K the unique subset of $[r]$ of cardinality d such that $S \in \Omega_K^o(E_\bullet(V)) \subseteq \text{Gr}(d, V)$. Set $L = \{i_a \mid a \in K\}$. Then, $S \in \Omega_L^o(E_\bullet) \subseteq \text{Gr}(d, W)$.

Lemma 2.3. Let V be a r -dimensional subspace of an n -dimensional vector space W , and $\mathcal{E} \in \text{Fl}(W)^s$. Let I^1, \dots, I^s be the unique subsets of $[n]$ each of cardinality r such that $V \in \cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$. Suppose $S \subseteq V$ is a d -dimensional vector subspace. Let K^1, \dots, K^s be the unique subsets of $[r]$ each of cardinality d such that $S \in \cap_{j=1}^s \Omega_{K^j}^o(E_\bullet^j(V))$.

Then

$$\dim(S, V, \mathcal{E}(V)) - \dim(S, W, \mathcal{E}) = \sum_{j=1}^s \sum_{a \in K^j} (n - r + a - i_a^j) - d(n - r).$$

2.3. Tangent spaces. Let $V \in \text{Gr}(r, W)$. If $\mathcal{E} \in \text{Fl}(W)^s$ and I^1, \dots, I^s are the unique subsets of $[n]$ each of cardinality r such that $V \in \cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$, then the tangent space at V to the scheme theoretic intersection $\cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$ is given by (see [GH])

$$(2.1) \quad \{\phi \in \text{Hom}(V, W/V) \mid \phi(E_a^j(V)) \subseteq E_{i_a^j - a}^j(W/V) \text{ for } a = 1, \dots, r, j = 1, \dots, s\}.$$

Definition 2.4. Let $\mathcal{I} = (I^1, \dots, I^s)$ be a s -tuple of subsets of $[n]$ of cardinality r each. Let V and Q be vector spaces of rank r and $n-r$ respectively and $(\mathcal{F}, \mathcal{G}) \in \text{Fl}(V)^s \times \text{Fl}(Q)^s$. Define

$$(2.2) \quad \text{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G}) = \bigcap_{j=1}^s \{\phi \in \text{Hom}(V, Q) \mid \phi(F_a^j) \subseteq G_{i_a^j - a}^j \text{ for } a = 1, \dots, r\}.$$

$$(2.3) \quad = \{\phi \in \text{Hom}(V, Q) \mid \phi(F_a^j) \subseteq G_{i_a^j - a}^j \text{ for } a = 1, \dots, r, j = 1, \dots, s\}$$

2.4. Stratification and Universal families. Let $\mathcal{I} = (I^1, \dots, I^s)$, V and Q be as before. Let $\mathcal{K} = (K^1, \dots, K^s)$ be a s -tuple of subsets of $[r]$ each of cardinality d . We consider the following “universal objects” (see [GH] for more details).

(A) Define $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$ to be the scheme over $\text{Fl}(V)^s \times \text{Fl}(Q)^s$ whose fiber over $(\mathcal{F}, \mathcal{G})$ is

$$(2.4) \quad \{\phi \in \text{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G}) \mid \text{rk}(\ker(\phi)) = d, \ker(\phi) \in \cap_{j=1}^s \Omega_{K^j}^o(F_\bullet^j)\}.$$

(B) Define $\mathfrak{U}_{\mathcal{K}}(V)$ to be the scheme over $\text{Fl}(V)^s$ whose fiber over $\mathcal{F} \in \text{Fl}(V)^s$ is $\cap_{j=1}^s \Omega_{K^j}^o(F_\bullet^j) \subseteq \text{Gr}(d, V)$.

Proposition 2.5. (1) $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$ and $\mathfrak{U}_{\mathcal{K}}(V)$ are smooth and irreducible schemes.
 (2) If $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K}) \neq \emptyset$, the natural morphism $p : \mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K}) \rightarrow \mathfrak{U}_{\mathcal{K}}(V)$ which maps $(\phi, \mathcal{F}, \mathcal{G})$ to $(\ker(\phi), \mathcal{F})$, is smooth and surjective.
 (3) The dimension of $\mathfrak{U}_{\mathcal{K}}(V)$ is $\dim \text{Fl}(V)^s + \dim \text{Gr}(d, r) - \sum_{j=1}^s \text{codim}(\omega_{K^j})$.
 (4) The dimension of $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$ is

$$\dim \mathfrak{U}_{\mathcal{K}}(V) + \dim \text{Fl}(Q)^s + \left\{ \sum_{j=1}^s \sum_{a \in K^j} (n - r + a - i_a^j) - d(n - r) \right\}.$$

3. THE SETTING FOR THE PROOF OF FULTON'S CONJECTURE

A subset $I \subseteq [n]$ of cardinality r also defines an irreducible representation $V_{\lambda(I)}$ of $\text{GL}(r)$, where $\lambda(I) = (\lambda_1 \geq \dots \geq \lambda_r)$ with $\lambda_a = n - r + a - i_a$ for $a = 1, \dots, r$.

Now, let I^1, \dots, I^s be subsets of $[n] = \{1, \dots, n\}$ each of cardinality r . Assume that

$$\sum_{j=1}^s \sum_{a=1}^r (n - r + a - i_a^j) = r(n - r).$$

Let $\lambda_j = \lambda(I^j)$ be the weights of the corresponding irreducible representations of $\mathrm{GL}(r)$.

Then it is well known that,

$$\prod_{j=1}^s \omega_{I^j} = \dim_{\mathbb{C}}[V_{\lambda_1} \otimes \dots \otimes V_{\lambda_s}]^{\mathrm{SL}(r)}[\text{class of a point}] \in H^*(\mathrm{Gr}(r, n)).$$

The above notation and assumptions will be kept fixed throughout this paper and will be called the Fixed Setting. There is a related space of parabolic vector spaces relevant to this setting:

3.1. Parabolic vector spaces. A parabolic vector space \tilde{V} is a 3-tuple (V, \mathcal{F}, w) , where V is a vector space, $\mathcal{F} \in \mathrm{Fl}(V)^s$ and w is a function

$$w : \{1, \dots, s\} \times \{1, \dots, \mathrm{rk}(V)\} \rightarrow \mathbb{Z}$$

such that if we let $w_l^j = w(j, l)$, the following holds for each $j = 1, \dots, s$:

$$w_1^j \geq w_2^j \geq \dots \geq w_{\mathrm{rk}(V)}^j.$$

An isomorphism between parabolic vector spaces of the same rank and weights w , (V, \mathcal{F}, w) and (T, \mathcal{G}, w) is an isomorphism $V \rightarrow T$ such that for any $j \in 1, \dots, s$ and $a < \mathrm{rk}(V)$ such that $w_a^j > w_{a+1}^j$, $\phi(F_a^j) = G_a^j$. So in reality one ignores the parts of the flags where the weights do not jump (we may similarly define morphisms between parabolic vector spaces, and create a corresponding abelian category).

Let $S \subseteq V$ be a non zero subspace of rank d . Let K^1, \dots, K^s be the unique subsets of $[\mathrm{rk}(V)]$ each of cardinality d such that $S \in \cap_{j=1}^s \Omega_{K^j}^o(F_{\bullet}^j)$. Define the parabolic slope

$$\mu(S, \tilde{V}) = \frac{\sum_{j=1}^s \sum_{a \in K^j} w_a^j}{d}.$$

A parabolic vector space \tilde{V} is said to be semistable if for each subspace $S \subseteq V$, $\mu(S, \tilde{V}) \leq \mu(V, \tilde{V})$.

Given the fixed setting we get a choice of weights for parabolic vector spaces. Here we consider parabolic vector spaces of the form (V, \mathcal{F}, w) with $\mathrm{rk}(V) = r$ and

$$w_a^j = \lambda_a^j = n - r + a - i_a^j, \quad j = 1, \dots, s, \quad a = 1, \dots, r.$$

3.2. Moduli spaces. Let

$$\mathcal{M} = \mathcal{M}(\mathcal{I}, r, n) = \mathrm{Proj} \bigoplus_{N=1}^{\infty} (V_{N\lambda_1}^* \otimes \dots \otimes V_{N\lambda_s}^*)^{\mathrm{SL}(r)}.$$

be the (projective and irreducible) moduli space of semistable parabolic vector spaces with the above weights. The proof of the properties below follows similar properties for parabolic vector bundles.

Let V be a r dimensional vector space. There is an open subset U of $FL(V)^s$ formed by points \mathcal{F} so that (V, \mathcal{F}, w) is semistable. There is a natural surjective map $U \rightarrow \mathcal{M}$, and there is a natural line bundle \mathcal{L} on \mathcal{M} obtained by descent via Kempf's theory (see [P]) of the natural line bundle $\tilde{\mathcal{L}} = \mathcal{L}_{\lambda_1} \otimes \dots \otimes \mathcal{L}_{\lambda_s}$ on $Fl(V)^s$ whose global sections are $V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_s}^*$. In fact, note that global sections (for example [T])

$$(3.1) \quad H^0(\mathcal{M}, \mathcal{L}) = H^0(FL(V)^s, \tilde{\mathcal{L}})^{SL(V)}.$$

Similarly,

$$(3.2) \quad H^0(\mathcal{M}, \mathcal{L}^N) = [V_{N\lambda_1}^* \otimes \dots \otimes V_{N\lambda_s}^*]^{SL(r)}.$$

3.3. Formulation of theorems as properties of \mathcal{M} . The saturation theorem of Knutson and Tao can therefore be formulated as: For any positive integer N ,

$$h^0(\mathcal{M}, \mathcal{L}) \neq 0 \Leftrightarrow h^0(\mathcal{M}, \mathcal{L}^N) \neq 0$$

Fulton's conjecture (theorem of Knutson, Tao and Woodward) can be formulated as : For any positive integer N ,

$$h^0(\mathcal{M}, \mathcal{L}) = 1 \Leftrightarrow h^0(\mathcal{M}, \mathcal{L}^N) = 1$$

In view of the ampleness of \mathcal{L} on \mathcal{M} , and the connectedness of \mathcal{M} we have a reformulation of saturation and Fulton's conjecture, the saturation statement is

$$\mathcal{M} \neq \emptyset \Rightarrow h^0(\mathcal{M}, \mathcal{L}) \neq 0.$$

Fulton's conjecture is then the statement

$$\mathcal{M} \neq \emptyset, \dim(\mathcal{M}) > 0 \Rightarrow h^0(\mathcal{M}, \mathcal{L}) \neq 1.$$

(This is applied to \mathcal{L} and \mathcal{L}^N , the different linearization $\tilde{\mathcal{L}}^N$ does not change \mathcal{M} but changes the basic line bundle on it to \mathcal{L}^N .)

3.4. The starting point. Assume $\mathcal{M} \neq \emptyset$, $\dim(\mathcal{M}) > 0$ and $h^0(\mathcal{M}, \mathcal{L}) = 1$, and we will show that this leads to a contradiction. We will indicate the starting point of the argument now. Let $\Theta \in h^0(\mathcal{M}, \mathcal{L})$ be the unique non-vanishing section (upto scalars). Since \mathcal{L} is ample and \mathcal{M} positive dimensional, **the zero set of Θ is non-empty**. therefore Θ vanishes at a point of the form (V, \mathcal{F}) which is **semistable**.

To "test" the assumption $h^0(\mathcal{M}, \mathcal{L}) = 1$, we will need a way of producing global sections of \mathcal{L} . In [I] we showed a way of producing all sections of $h^0(\mathcal{M}, \mathcal{L})$ via Schubert calculus.

3.5. Construction of sections of \mathcal{L} . We return to the notation of the fixed setting. Let Q be a vector space of rank $n - r$ and $\mathcal{G} \in \text{Fl}(Q)^s$. In [I] we showed that the pair (Q, \mathcal{G}) can be used to produce a section $\Theta(Q, \mathcal{G})$ of

$$H^0(\mathcal{M}, \mathcal{L}) = [V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_s}^*]^{\text{SL}(r)}$$

We briefly recall the description: The zero set of $\Theta(Q, \mathcal{G})$ on $FL(V)^s$ is the set of points (V, \mathcal{E}) for which the vector space $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{E}, \mathcal{H})$ (see Section 2.3) is non-zero (this condition is converted into a determinantal condition and hence a section of the desired bundle.) If W is an n -dimensional vector space, $\mathcal{F} \in \text{Fl}(W)^s$ generic and $\{V_1, \dots, V_m\} = \bigcap_{j=1}^s \Omega_{I_j}^o(F_{\bullet}^j)$, then the sections $\Theta(W/V_{\ell}, \mathcal{F}(W/V_{\ell}))$ give a basis for $[V_{\lambda_1}^* \otimes \dots \otimes V_{\lambda_s}^*]^{\text{SL}(r)}$ (in fact $\Theta(W/V_{\ell}, \mathcal{F}(W/V_{\ell}))$ vanishes at $(V_k, \mathcal{F}(V_k))$ if and only if $\ell \neq k$) (see Section 2.2).

4. RETURN TO THE PROOF OF FULTON'S CONJECTURE

We now return to the situation at the end of Section 3.4. Let Q be a vector space of rank $n - r$. Let $\Theta \in H^0(\mathcal{M}, \mathcal{L})$ be the unique non-zero section (upto scalars).

Let $\mathfrak{Z} \subset FL(V)^s$ be the closure of an irreducible component of the zero set of Θ which contains a semistable point (\mathfrak{Z} is to be fixed once and for all). Recall that the set of semistable is open in any family (and by Equation 3.1, we can consider sections of \mathcal{L} as invariant sections of $\tilde{\mathcal{L}}$ on $FL(V)^s$).

Let $(\mathcal{E}, \mathcal{H})$ be a generic point of $\mathfrak{Z} \times \text{Fl}(Q)^s$. We know that section $\Theta(Q, \mathcal{H})$ is the unique non-zero section of \mathcal{L} upto scalars. Therefore $\Theta(Q, \mathcal{H})$ vanishes at (V, \mathcal{E}) , in other words $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{E}, \mathcal{H})$ is a non-zero vector space. Let ϕ be a generic element of $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{E}, \mathcal{H})$. Let $S = \ker(\phi)$, $d = \text{rk}(S)$ ($d = 0$ is possible!), and let $\mathcal{K} = (K^1, \dots, K^s)$ be the unique s -tuple of subsets of $[r]$ each of cardinality d such that $S \in \bigcap_{j=1}^s \Omega_{K^j}^o(E_{\bullet}^j)$.

Proposition 4.1. *Let (V, \mathcal{F}) be such that $\bigcap_{j=1}^s \Omega_{K^j}^o(F_{\bullet}^j) \neq \emptyset$. Then $\mathcal{F} \in \mathfrak{Z}$.*

Proof. We consider the spaces $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$ and $\mathfrak{U}_{\mathcal{K}}(V)$ from [GH] (and recalled in Section 2.4 of this paper):

Let X be the nonempty open subset of $\text{Fl}(Q)^s$ formed by points \mathcal{G} so that $\theta(Q, \mathcal{G}) \neq 0$. We claim that if

$$Y = \{(V, \mathcal{F}, \mathcal{G}) \in \mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K}) \mid \mathcal{G} \in X\}$$

and $(V, \mathcal{F}, \mathcal{G}) \in Y$ then Θ vanishes at (V, \mathcal{F}) . This is clear because $\Theta(Q, \mathcal{G})$ vanishes at (V, \mathcal{F}) and Θ is a multiple of $\Theta(Q, \mathcal{G})$.

Clearly Y is non-empty (by assumption!) and dense in the irreducible $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$. Therefore Θ vanishes at all points of the form (V, \mathcal{F}) for which there is a point of the form $(V, \mathcal{F}, \mathcal{G}) \in \mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$. The surjectivity of $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K}) \rightarrow \mathfrak{U}_{\mathcal{K}}(V)$ therefore assures us that Θ vanishes at (V, \mathcal{F}) if there exists a point of the form $(S, \mathcal{F}) \in \mathfrak{U}_{\mathcal{K}}(V)$. Since $\mathfrak{U}_{\mathcal{K}}(V)$ is irreducible, the proof is complete. \square

Let $\mathfrak{A} \subseteq \mathrm{Fl}(V)^s$ be the closure of the image of $\mathfrak{U}_{\mathcal{K}}(V)$. By Proposition 4.1 $\mathfrak{A} \subseteq \mathfrak{Z}$ (note that \mathfrak{A} being the closure of the image of $\mathfrak{U}_{\mathcal{K}}(V)$ is irreducible). By assumption $\mathfrak{Z} \subseteq \mathfrak{A}$. Hence

Lemma 4.2. $\mathfrak{A} = \mathfrak{Z}$.

Let R, T be vector spaces of dimension d and $r - d$ respectively, and U a nonempty open subset of $\mathrm{Fl}(R)^s \times \mathrm{Fl}(T)^s$ which is stable under $\mathrm{GL}(R)^s \times \mathrm{GL}(T)^s$ (so that one has canonically defined open subsets of $\mathrm{Fl}(R')^s \times \mathrm{Fl}(T')^s$ for any vector spaces R' and T' of ranks d and $r - d$ respectively).

Proposition 4.3. *There exists a nonempty open subset \tilde{U} of $\mathfrak{Z} \times \mathrm{Fl}(Q)^s$ such that for $(\mathcal{F}, \mathcal{G}) \in \tilde{U}$,*

- (a) *The intersection $\cap_{j=1}^s \Omega_{K^j}^o(F_{\bullet}^j)$ is equidimensional of dimension $\dim \mathfrak{U}_{\mathcal{K}}(V) - \dim \mathfrak{Z}$.*
- (b) *If ϕ is a general element of $\mathrm{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$, and $S = \ker(\phi)$ then the induced pair of flags $(\mathcal{F}(S), \mathcal{F}(V/S))$ “is a point of U ”.*
- (c) *The rank of $\mathrm{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$ is*

$$\dim \mathfrak{U}_{\mathcal{K}}(V) - \dim \mathfrak{Z} + \left\{ \sum_{j=1}^s \sum_{a \in K^j} (n - r + a - i_a^j) - d(n - r) \right\}.$$

Proof. Item (a) follows from generic flatness of $\mathfrak{U}_{\mathcal{K}}(V) \rightarrow \mathfrak{A} = \mathfrak{Z}$.

Let $W_1 \subseteq \mathfrak{U}_{\mathcal{K}}(V)$ be the non-empty open subset of points (S, \mathcal{F}) such that $(\mathcal{F}(S), \mathcal{F}(V/S))$ is a point of U . Let W_2 be the inverse image of W_1 in $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K})$. Let \tilde{U} be an open subset of $\mathfrak{Z} \times \mathrm{Fl}(Q)^s$ so that the map $[\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K}) - W_2] \rightarrow \mathfrak{Z} \times \mathrm{Fl}(Q)^s$ is flat over \tilde{U} . This proves (b).

The fiber dimension of $\mathfrak{H}_{\mathcal{I}}(V, Q, \mathcal{K}) \rightarrow \mathfrak{Z} \times \mathrm{Fl}(Q)^s$ is easily seen to be

$$\dim \mathfrak{U}_{\mathcal{K}}(V) - \dim \mathfrak{Z} + \left\{ \sum_{j=1}^s \sum_{a \in K^j} (n - r + a - i_a^j) - d(n - r) \right\}.$$

This proves (c). □

4.0.1. The conclusion of the proof of Fulton's conjecture. Let $(\mathcal{F}, \mathcal{G})$ be a general point of $\mathfrak{Z} \times \mathrm{Fl}(Q)^s$ as in Proposition 4.3. Let ϕ be a general point of $\mathrm{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$, $S = S(1) = \ker(\phi)$. Then the induced flags $(S, \mathcal{F}(S))$ and $(V/S, \mathcal{F}(V/S))$ can be assumed to be general and in mutually general position. This follows from proposition 4.3.

Set $L^j(1) = K^j$ for $j = 1, \dots, s$. Now suppose that $\cap_{j=1}^s \Omega_{K^j}^o(F_{\bullet}^j)$ is positive dimensional at $S = S(1)$.

If $\psi^{(1)}$ is a generic element in the tangent space of $\cap_{j=1}^s \Omega_{K^j}^o(F^j) \subseteq \mathrm{Gr}(d, V)$ at S , we can view $\psi^{(1)}$ as a map $S \rightarrow V/S$ (the tangent space to $\mathrm{Gr}(d, V)$ at S is $\mathrm{Hom}(S, V/S)$). Let $S^{(2)}$ be the kernel of $\psi^{(1)}$ and assume that $S^{(2)} \in \cap_{j=1}^s \Omega_{L^j(2)}^o(F^j(S)) \subseteq \mathrm{Gr}(d, S)$.

We obtain a sequence of inclusions

$$S^{(h)} \subsetneq S^{(h-1)} \subsetneq \dots \subsetneq S^{(1)} = S \subsetneq V$$

inductively as follows. Assume

$$S^{(\ell)} \in \cap_{j=1}^s \Omega_{L^j(\ell)}^o(F^j(S^{(\ell-1)})) \subseteq \text{Gr}(d, S^{(\ell-1)}).$$

If this intersection is 0 dimensional at $S^{(\ell)}$ then the process stops at $h = \ell$. If it is positive dimensional at S , let $\psi^{(\ell)}$ be the generic element of the tangent space at $S^{(\ell)}$ of

$$\cap_{j=1}^s \Omega_{L^j(\ell)}^o(F^j(S^{(\ell-1)})) \subseteq \text{Gr}(d, S^{(\ell-1)}).$$

View $\psi^{(\ell)}$ as a map $S^{(\ell)} \rightarrow S^{(\ell-1)}/S^{(\ell)}$ and define $S^{(\ell+1)} = \ker \psi^{(\ell)}$. And continue with the “recursion”. This procedure starting from S will be called the “tangent space method”.

For $u = 1, \dots, h$, let d_u be the rank of $S^{(u)}$, $\mathcal{J}(u) = (J^1(u), \dots, J^s(u))$ the unique s -tuple of subsets of $[r]$ each of cardinality d_u such that $S^{(u)} \in \cap_{j=1}^s \Omega_{J^j(u)}^o(F_{\bullet}^j) \subseteq \text{Gr}(d_u, V)$, By Lemma 2.2 (applied to $S^{(u)} \subseteq S \subseteq V$ and $\mathcal{F} \in \text{Fl}(V)^s$)

$$(4.1) \quad J^j(u) = \{k_b^j \mid b \in L^j(u)\}$$

We claim

Proposition 4.4. (i) $\dim \mathfrak{U}_{\mathcal{K}}(V) - \dim \mathfrak{Z} = \dim \cap_{j=1}^s \Omega_{K^j}^o(F_{\bullet}^j)$ is less than or equal to (in fact equal to, we will not need this)

$$\dim(S, V, \mathcal{F}) + \dim(S^{(h)}, S, \mathcal{F}(S)) - \dim(S^{(h)}, V, \mathcal{F}).$$

(ii)

$$\begin{aligned} & \dim(S, V, \mathcal{F}) + \left\{ \sum_{j=1}^s \sum_{a \in K^j} (n - r + a - i_a^j) - d_u(n - r) \right\} \\ & \leq \dim(S^{(h)}, V, \mathcal{F}) - \dim(S^{(h)}, S, \mathcal{F}(S)) + \left\{ \sum_{j=1}^s \sum_{a \in J^j(h)} (n - r + a - i_a^j) - d_h(n - r) \right\} \end{aligned}$$

From (i) and (ii), and Proposition 4.3, we conclude that the rank of $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$ is no greater than

$$(4.2) \quad \sum_{j=1}^s \sum_{a \in J^j(h)} (n - r + a - i_a^j) - d_h(n - r)$$

The semistability of (V, \mathcal{F}, w) we obtain that Expression 4.2 is ≤ 0 and hence the geometric proof of Fulton’s conjecture would be complete once Proposition 4.4 is proved.

Proof. (Of Proposition 4.4) The dimension of $\dim \cap_{j=1}^s \Omega_{K^j}^o(F_{\bullet}^j)$ is no more than the dimension of its tangent space at S which is $\text{Hom}_{\mathcal{K}}(S, V/S, \mathcal{F}(S), \mathcal{F}(V/S))$ and using the main theorem in [GH] (and Lemma 2.3) we find that (i) holds.

Item (ii) follows from the filtration lemma in [GH] (recalled below) where we apply it with $\eta_u = \phi \circ \psi^{(u)}$. \square

4.1. The Filtration Lemma. For a vector space W of rank n , define $B(W) \subseteq \text{Fl}(W)^s$ to be the largest Zariski open subset of $\text{Fl}(W)^s$ satisfying the following property: If $\mathcal{E} \in B(W)$ and $\mathcal{I} = (I^1, \dots, I^s)$ a s -tuple of subsets of $[n]$ each of the same cardinality r , then every irreducible component of the intersection $\cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$ (which is possibly empty) is proper. By Kleiman's transversality theorem [K], it follows that $B(W)$ is nonempty.

Lemma 4.5. *Consider a 5-tuple of the form $(V, Q, \mathcal{F}, \mathcal{G}, \mathcal{I})$ where V and Q are non-zero vector spaces of ranks r and $n - r$ respectively, $\mathcal{I} = (I^1, \dots, I^s)$ a s -tuple of subsets of $[n]$ each of cardinality r and $\mathcal{G} \in B(Q)$.*

Suppose in addition that we are given a filtration by vector subspaces

$$(4.3) \quad S^{(h)} \subsetneq S^{(h-1)} \subsetneq \dots \subsetneq S^{(1)} \subsetneq S^{(0)} = V$$

and injections (of vector spaces) from the graded quotients $\eta_u : S^{(u)}/S^{(u+1)} \hookrightarrow Q$ for $u = 0, \dots, h-1$ such that for $j = 1, \dots, s$ and $a = 1, \dots, r$, (where we write η_u again for the composite $S^{(u)} \rightarrow S^{(u)}/S^{(u+1)} \xrightarrow{\eta_u} Q$)

$$\eta_u(S^{(u)} \cap F_a^j) \subseteq G_{i_a^j - a}^j$$

Then, letting d_u be the rank of $S^{(u)}$, $\mathcal{J}(u) = (J^1(u), \dots, J^s(u))$ the unique s -tuple of subsets of $[r]$ each of cardinality d_u such that $S^{(u)} \in \cap_{j=1}^s \Omega_{J^j(u)}^o(F_\bullet^j) \subseteq \text{Gr}(d_u, V)$ and

$$\begin{aligned} & \dim(S, V, \mathcal{F}) + \left\{ \sum_{j=1}^s \sum_{a \in K^j} (n - r + a - i_a^j) - d_u(n - r) \right\} \\ & \leq \dim(S^{(h)}, V, \mathcal{F}) - \dim(S^{(h)}, S, \mathcal{F}(S)) + \left\{ \sum_{j=1}^s \sum_{a \in J^j(h)} (n - r + a - i_a^j) - d_h(n - r) \right\} \end{aligned}$$

APPENDIX A. RESUME OF RESULTS IN [GH]

Let $\mathcal{I} = (I^1, \dots, I^s)$ be a s -tuple of subsets of $[n]$ of cardinality r each and W an n -dimensional vector space. Let us ask the following questions:

- (Q1) For generic $\mathcal{E} \in \text{Fl}(W)^s$ is $\cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$ empty?
- (Q2) Let $\mathfrak{A} \subseteq \text{Fl}(V)^s$ be the set of \mathcal{E} such that $\cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$ is non-empty. For generic (\mathfrak{A} is the image of $\mathfrak{U}_{\mathcal{I}}(W)$ and is hence irreducible) $\mathcal{E} \in \mathfrak{A}$ what is then the dimension of (each irreducible component of) $\cap_{j=1}^s \Omega_{I^j}^o(E_\bullet^j)$?

Let V and Q be vector spaces of rank r and $n - r$ respectively and $(\mathcal{F}, \mathcal{G}) \in \text{Fl}(V)^s \times \text{Fl}(Q)^s$ a generic point. Then, to answer these questions, according to [GH] one needs to proceed as follows. The answer for (Q2) is the rank of $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$. If this rank equals the expected dimension $[\dim(\text{Gr}(r, n)) - \sum_{j=1}^s \text{codim}(\omega_{I^j})]$, then the answer to (Q1) is affirmative (and vice-versa).

Theorem A.1. *There exists a filtration by vector subspaces obtained by the “tangent space” method*

$$(A.1) \quad S^{(h)} \subsetneq S^{(h-1)} \subsetneq \dots \subsetneq S^{(1)} \subsetneq S^{(0)} = V$$

and injections (of vector spaces) from the graded quotients $\eta_u : S^{(u)}/S^{(u+1)} \hookrightarrow Q$ for $u = 0, \dots, h-1$ such that the following property is satisfied: For $u = 1, \dots, h$, let d_u be the rank of $S^{(u)}$, $\mathcal{J}(u) = (J^1(u), \dots, J^s(u))$ the unique s -tuple of subsets of $[r]$ each of cardinality d_u such that $S^{(u)} \in \cap_{j=1}^s \Omega_{J^j(u)}^o(F_\bullet^j) \subseteq \text{Gr}(d_u, V)$, then

$$(i) \quad \dim(S^{(h)}, V, \mathcal{F}) = 0.$$

(ii) For $u = 0, \dots, h-1$, $j = 1, \dots, s$ and $a = 1, \dots, r$, (where we write η_u again for the composite $S^{(u)} \rightarrow S^{(u)}/S^{(u+1)} \xrightarrow{\eta_u} Q$)

$$\eta_u(S^{(u)} \cap F_a^j) \subseteq G_{i_a^j - a}^j$$

(iii) The vector space $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$ is of rank (the second term of the expression below is the same as the quantity appearing in Inequality $(\dagger_{\mathcal{J}(h)}^{\mathcal{I}})$)

$$(A.2) \quad [\dim(\text{Gr}(r, n)) - \sum_{j=1}^s \text{codim}(\omega_{I^j})] + \left\{ \sum_{j=1}^s \sum_{a \in J^j(h)} (n - r + a - i_a^j) - d_h(n - r) \right\}.$$

Remark A.2. *The filtration is constructed in the course of the proof of this theorem. Here we start with a generic element of $\text{Hom}_{\mathcal{I}}(V, Q, \mathcal{F}, \mathcal{G})$, let $S = \ker(\phi) \subset V$ and apply the tangent space method to it.*

REFERENCES

- [GH] P. Belkale. Geometric Proofs of Horn and Saturation conjectures. math AG/0208107, To appear in the Journal of Algebraic Geometry.
- [QH] P. Belkale, Quantum generalization of Horn and Saturation conjectures. preprint, math.AG/0303013.
- [I] P. Belkale. Invariant theory of $\text{GL}(n)$ and intersection theory of Grasmannians. International Math Research Notices, Vol 2004, no.69, 3709–3721.
- [F1] W. Fulton. Eigenvalues of Majorized Hermitian Matrices and Littlewood-Richardson Coefficients. Lin. Alg. Appl 319(2000) 23–36.
- [IT] W. Fulton. Intersection Theory. Springer-Verlag Berlin 1998.
- [F2] W. Fulton. Young Tableaux. London Math Society, Student Texts, 35, 1997.
- [H] R. Hartshorne Algebraic Geometry. Graduate texts in mathematics 52, Springer-Verlag, 1977.
- [K] S.L. Kleiman. The transversality of a general translate. Compositio Math. 38, (1974), 287–297.
- [KT] A. Knutson, T. Tao. The Honeycomb model of $\text{GL}_n(C)$ tensor products. I. Proof of the Saturation conjecture. J. Amer. Math. Soc. 12(1999) no.4, 1055–1090.
- [KTW] A. Knutson, T. Tao, C. Woodward The Honeycomb model of $\text{GL}_n(C)$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson Cone. To appear in Journal of American Math Society, math.CO/0107011.
- [P] C. Pauly, Espaces de modules de fibrés paraboliques et blocs conformes. Duke Math. J. 84 (1996), no. 1, 217–235.
- [T] C. Teleman, The quantization conjecture revisited. Ann. of Math. (2) 152 (2000), no. 1, 1–43.

DEPARTMENT OF MATHEMATICS, UNC-CHAPEL HILL, CB #3250, PHILLIPS HALL, CHAPEL HILL,
NC 27599

E-mail address: belkale@email.unc.edu